1 Lagrange Multipliers

This handout was inspired by Evan Chen's handout. This handout comes along with my video on Lagrange Multipliers which also gives a visually intuitive explanation on how Lagrange Multipliers work.

§1.1 Theory

Calculus is not required for math olympiads at any level, i.e. every problem has a solution that completely avoids calculus methods. One often does not get any problems when simply taking derivatives to check whether a function is increasing, convex, etc., but when using more advanced methods graders tend to get very picky with details, so even doing one sign error may result in getting 0 points instead of 7. This section will introduce Lagrange Multipliers, a method that can often optimize a function that has a restriction on its variables.

Theorem 1.1.1: Lagrange Multipliers

Let $U \subset \mathbb{R}^n$ be an open subset, and let $f : U \longrightarrow \mathbb{R}$ and $g : U \longrightarrow \mathbb{R}$ be two functions with continuous first derivatives. Define the constraint set

$$S = \{x \in U \mid g(x) = c\}$$

Suppose $x' \in S$ is a local maximum of f along S. Then either $\nabla g(x') = 0$ at that point, or for some real number λ

$$\nabla f(x') = \lambda \nabla g(x')$$

So far, this does not tell us how to actually find a global maximum/minimum, as we usually want to. But notice that there does not even need to be a global maximum/minimum. However, we can often transform the condition in the theorem statement that actually helps us getting something out of it.

Lemma 1.1.1: Compact Sets induce Maximums — Let K be a compact set and let $f : K \longrightarrow \mathbb{R}$ be a continuous function. Then f achieves a global maximum and a global minimum over K.

This means that we will usually make U compact by showing that it is bounded and take its closure $\overline{U}$. This then guarantees that we could find global extrema over U. However, we want to find a global extremum over the constraint set, so we need the following lemma.

Lemma 1.1.2: Pre-Images are Closed Sets — Let $g : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a continuous function. Then the set

$$C = \{x \in \mathbb{R}^n \mid g(x) = c\}$$

is closed for every real number c.

Now we can consider the function $f : \overline{U} \longrightarrow \mathbb{R}$ subject to the constraint set, which we also need to be compact. By our second lemma, the set

$$C = \{x \in \mathbb{R}^n \mid g(x) = c\}$$

is closed. We now define

$$S = U \cap C = \{x \in U \mid g(x) = c\}$$

and

$$\overline{S} = \overline{U} \cap C = \{x \in \overline{U} \mid g(x) = c\}$$

As $\overline{U}$ and C are closed, $\overline{S}$ has to be closed as well. If one of them is also bounded, $\overline{S}$ is bounded, and f achieves a global maximum over $\overline{S}$. But since we only want to consider $x \in U$, we have to consider two cases.

1. $x \in \overline{U} \backslash U$: This has to be done manually.
2. $x \in U$: Here we can now use Lagrange Multipliers.

After this is done, we can use Lagrange Multipliers: First of all we have to check whether $\nabla g(x)$ can be 0. If this is not the case, we can check all critical points and find the global maximum or minimum. Here we can see why Lagrange Multipliers only work for open sets U: if it where closed, then we could have a global extremum at the interval endpoint, where the gradiant does not necessarily have to be 0. But since we eliminated that case, we are good to go. The following is now the standard procedure when solving a problem with Lagrange Multipliers.

1. State/show that f and g have continuous first derivatives.
2. Consider the closure $\overline{U}$ of the function f you want to optimize and consider the set $C = \{x \in \mathbb{R}^n \mid g(x) = c\}$, which has to be closed.
3. Define S and $\overline{S}$.
4. Show that $\overline{S}$ is bounded and hence compact. State that now f has to achieve a maximum/minimum over S.
5. Check the boundary region and conclude that the points in there cannot be the extremum you want to find. Thus f must achieve a global maximum over S.
6. Check that $\nabla g(x) \neq 0$.
7. Check all critical points. If you want to find a maximum (resp minimum), the critial point with the largest (resp. smallest) value must be it.

§1.2 Examples

Example 1.2.1: Korea 2019

Let x, y, z be real numbers such that $x^2 + y^2 + z^2 = 174$. Find the difference between the maximum and the minimum of the following expression:

$$x + y + z - xy - yz - zx.$$

Proof. Clearly, $x, y, z \in [-\sqrt{147}, \sqrt{147}] =: U$. Define the function

$$f : U \longrightarrow \mathbb{R}, f(x, y, z) = x + y + z - xy - yz - zx$$

and the function

$$g(x, y, z) = x^2 + y^2 + z^2$$

which are both clearly differentiable with continuous first derivatives. Set $C = \{r \in \mathbb{R}^3 \,|\, g(r) = 147\}$. We want to optimize f over the set

$$S = U \cap C = \{r \in U \,|\, g(r) = 147\}$$

As U and C are closed and U is bounded, S is compact, so f achieves a global maximum and minimum over S. We may now use Lagrange Multipliers. Clearly $\nabla g(x) = \langle 2x, 2y, 2z \rangle \neq 0$, so there exists a real number λ such that

$$\begin{cases} x^2 + y^2 + z^2 = 147 \\ 1 - y - z = \lambda \cdot 2x \\ 1 - z - x = \lambda \cdot 2y \\ 1 - x - y = \lambda \cdot 2z \end{cases}$$

Subtracting the third from the second gives gives $x - y = 2\lambda(x - y)$. If $2\lambda \neq 1$ than $x = y = z$, so $3x^2 = 147 \longrightarrow x = y = z = \pm 7$, giving the critical points -168 and -126. Otherwise, we have $\lambda = \frac{1}{2}$, so $x + y + z = 1$. Then, we have

$$1 = (x + y + z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx) = 147 + 2(xy + yz + zx)$$

or $-xy - yz - xz = 73$, meaning $f(x, y, z) = x + y + z + 73 = 74$. Since those are the three critical points, we have that $\max(f) - \min(f) = 74 - (-168) = 242$. □

Example 1.2.2: Germany, Round 3, 2015/5

et x, y, z be real numbers such that $x + y + z = 0$ and all numbers are at most 1. Prove that

$$x^2 + y^2 + z^2 - xy - yz - zx \leqslant 9$$

and determine all equality cases.

Proof. It is clear that $x, y, z \geqslant -2$, because otherwise the sum condition gets violated. Define the function $f : U = [-2, 1]^3 \longrightarrow \mathbb{R}$ with $f(x, y, z) = x^2+y^2+z^2-xy-yz-zx$ and $g(x, y, z) = x+y+z = 0$. It is easy to see that both functions are differentiable with continuous first partial derivatives. Define the set

$$C = \{x \in \mathbb{R}^3 \,|\, g(x) = 0\}$$

which is well-known to be closed. We want to maximize f along the set

$$S = U \cap C = \{x \in U \,|\, g(x) = 0\}$$

As U and C are closed and U is furthermore bounded, S is compact and f achieves a global maximum over S. Let $U' = (-2, 1)^3$. We now have two cases.

1. $(x, y, z) \in U \backslash U'$, i.e $x, y, z \in \{-2, 1\}$. In that case, exactly one variable must be -2 and the other two must be 1 in order for $x + y + z = 0$ to be satisfied. It is easy to check that the inequality is satisfied in this case, even equality is achieved.
2. $(x, y, z) \in (-2, 1)^3$. For that we may use Lagrange multipliers.

First of all $\nabla g(x) = \langle 1, 1, 1 \rangle \neq 0$. Therefore, we can introduce a real number λ such that

$$\begin{cases} x + y + z = 0 \\ 2x - y - z = \lambda \cdot 1 \\ 2y - z - x = \lambda \cdot 1 \\ 2z - x - y = \lambda \cdot 1 \end{cases}$$

so $x = y = z = 0$. This critical point is the only other point besides the three points from the first case where a global maximum could occur, but (x, y, z) is clearly not a maximum, so we are done. □

§1.3 Problems

Problem 1.1 (Canada MO 1999/5)**.** Prove that for nonnegative real numbers a, b, c with $a+b+c=1$ we have

$$a^2b + b^2c + c^2a \leqslant \frac{4}{27}.$$

§1.4 Solutions

Problem 1.1 (Canada MO 1999/5)**.** Prove that for nonnegative real numbers a, b, c with $a+b+c=1$ we have

$$a^2b + b^2c + c^2a \leqslant \frac{4}{27}.$$

Proof. Define the function $f : U = [0,1]^3 \longrightarrow \mathbb{R}$ with $f(a,b,c) = a^2b + b^2c + c^2a$ and the constraint function $g(a,b,c) = a+b+c$ with the constraint set $C = \{x \in \mathbb{R}^n \mid g(x) = 1\}$. Clearly both functions have continuous first derivatives. We want to maximize the function f over the set

$$S = U \cap C = \{x \in U \mid g(x) = 1\}$$

Since U and C are closed and U is bounded, S is compact and f achieves a global maximum over S. Consider the set

$$S' = \{x \in U' = [0,1]^3 \backslash (0,1)^3 \mid g(x) = 1$$

. If one variable is 0, say c, then it is enough to show that $a^2b \leqslant \frac{4}{27}$, or $a^2(1-a) = a^2 - a^3 \leqslant \frac{4}{27}$ for $0 \leqslant a \leqslant 1$, which is easy by standard calculus methods. If two are 0, then f is 0. If one is 1, we are also done. The other cases clearly don't satisfy the constraint. Thus the maximum of f along U' is $\frac{4}{27}$. It suffices to prove that f does not succeed that value in $(0,3)$. Clearly, $\nabla g(x) = \langle 1,1,1 \rangle \neq 0$, so we may introduce Lagrange Multipliers.

$$\begin{cases} a+b+c=1 \\ 2ab = \lambda \\ 2bc = \lambda \\ 2ca = \lambda \end{cases}$$

We first set $2ab = \lambda = 2bc \iff a = c$. Similarly we get $a = b$, so $a = b = c$. Thus the only point in S where f could achieve a global maximum is $a = b = c = \frac{1}{3}$, for which we get $f(a,b,c) = \frac{4}{27}$. This completes the proof. □