Germany 2020, Grade 10, Round 3/6

Joel Gerlach

June 2022

Let a sequence of positive real numbers $a_1, a_2, a_3, \dots$ with $a_{59} = 59$ be given, such that $a_{m \cdot n} = a_m \cdot a_n$ and $a_m \leq a_n$ holds for all positive integers m, n with $m \leq n$.

Show that $a_{2019} = 2019$.

Proof. We have $a_{59\cdot k} = a_{59} \cdot a_k = 59 \cdot a_k$. Assume ftsoc that $a_2 = 1$. Then we would have $a_{118} = a_2 \cdot a_{59} = 59$ and thus $a_{59} = \ldots = a_{118} = 59$. This would also imply that $a_{2n} = a_2 \cdot a_n = a_n$. But this leads to a contradiction as this would imply $a_1 = a_2 = a_4 = a_8 = \ldots = a_{64} = 1 = 59$. Otherwise assume that $a_2 \geq 3$. Then $3481 = a_{3481} = a_{59\cdot 59} \geq a_{2048} = a_2^6 \geq 3^{11} \geq 100000$, a contradiction. It is easy to get $a_1 = 1$. We now have the following claim.

Claim 1 — $a_n < n + 1$.

Proof of Claim 1. We proceed by contradiction. Therefore, assume that $a_n \geq n + 1$. Choose j large enough such that for some k we have $59^j \geq n^k$ but $(n+1)^k > 59^j$. This gets us

$$a_{59^j} = (a_{59})^j = 59^j \geq a_{n^k} = (a_n)^k \geq (n+1)^k > 59^j,$$

a contradiction. □

Claim 2 — $a_n < a_{n+1}$

Proof of Claim 2. It suffices to check $a_n = a_{n+1}$ as $a_{n+1} < a_n$ is forbidden by the problem statement. Let $a_n = a_{n+1} = k$. We then have,

$$a_{n\cdot n} = a_n \cdot a_n \cdot a_{n+1} = a_{n\cdot(n+1)} = a_{n\cdot(n+1)}$$

And thus $a_n = a_{n+1} = \ldots = a_{n(n+1)}$. Repeating this for the last two of these equal terms we can eventually make all terms greater than a_n equal to each other, as long as $n \leq 59$, we actually force all $a_k = 59, k \geq n$, as $a_{59} = 59$ is given, which eventually leads to a contradiction as say $a_{59\cdot 59} = 59 \cdot 59 \neq 59$. Therefore we have $a_n > n - 1$ for all $59 \geq n \geq 1$. Similarly, if $a_n = a_{n+1} = k, n > 59$, we have to eventually make all terms equal to k, but if we choose j large enough and so that $59^j \neq k$, then $a_{59^j} = 59^j \neq k$, which is again a contradiction. □

Combining the two claims we get that $a_n = n$ for all n, which also implies that desired result. □